ORF 522
Linear Optimization

Lecture 17

Quadratic Programming

The Markowitz Model for Portfolio Optimization

What is Quadratic Programming?

Objective is convex quadratic.

Constraints **must be** linear.

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

Without loss of generality, the matrix Q of quadratic coefficients may be assumed symmetric.

The matrix Q must be positive semidefinite. Any of the following conditions characterize positive semidefiniteness:

- $\xi^T Q \xi \geq 0$ for all ξ.
- All eigenvalues of Q are nonnegative.
- There exists a matrix F such that $Q = F^T F$.

Markowitz Shares the 1990 Nobel Prize

Mirrors — Search — Site-map — Timeline

Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR'S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called, *Capital Asset Pricing Model* (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary
Financial markets serve a key purpose in a modern market economy by allocating productive resources among various areas of production. It is to a large extent through financial markets that saving in different sectors of the economy is transferred to firms for investments in buildings and machines. Financial markets also reflect firms' expected prospects and risks, which implies that risks can be spread and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry Markowitz who developed a theory for households' and firms' allocation of financial assets under uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally invested in assets which differ in regard to their expected return and risk, and thereby also how risks can be reduced.

Let's describe the Markowitz model...

Historical Data

Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

Notation: $R_j(t) =$ return on investment j in time period t.

Risk vs. Reward

Reward—estimated using historical means:

$$\text{reward}_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t).$$

Risk—estimated using historical variances:

$$\text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} \left(R_j(t) - \text{reward}_j\right)^2 .$$

Hedging

Investment A: up 20%, down 10%, equally likely.

Investment B: up 20%, down 10%, equally likely.

Correlation: up years for A are down years for B and vice versa.

Portfolio—half in A, half in B: up 5% every year!

Portfolios

Fractions: x_j = fraction of portfolio to invest in j.

Portfolio's Historical Returns:

$$R(t) = \sum_j x_j R_j(t)$$

Portfolio's Reward:

$$\text{reward}(x) = \frac{1}{T}\sum_{t=1}^{T} R(t) = \frac{1}{T}\sum_{t=1}^{T}\sum_j x_j R_j(t)$$

Portfolio's Risk:

$$\begin{aligned}
\text{risk}(x) &= \frac{1}{T}\sum_{t=1}^{T}\left(R(t) - \text{reward}(x)\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j x_j R_j(t) - \frac{1}{T}\sum_{s=1}^{T}\sum_j x_j R_j(s)\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j x_j \left(R_j(t) - \frac{1}{T}\sum_{s=1}^{T} R_j(s)\right)\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j x_j (R_j(t) - \text{reward}_j)\right)^2
\end{aligned}$$

Note: risk(x) is quadratic in the x_j's.

The Markowitz Model

Decision Variables: the fractions x_j.

Objective: maximize return, minimize risk.

Fundamental Lesson: can't simultaneously optimize two objectives.

Compromise: maximize a combination of reward and risk:

$$\text{reward}(x) - \mu\,\text{risk}(x)$$

Parameter μ is called risk aversion parameter.

$0 \le \mu < \infty$

Large value for μ puts emphasis on risk minimization.

Small value for μ puts emphasis on reward maximization.

Constraints:

$$\begin{aligned} \sum_j x_j &= 1 \\ x_j &\geq 0 \qquad \text{for all investments } j \end{aligned}$$

Quadratic Program:

Constraints are linear.

Objective is quadratic.

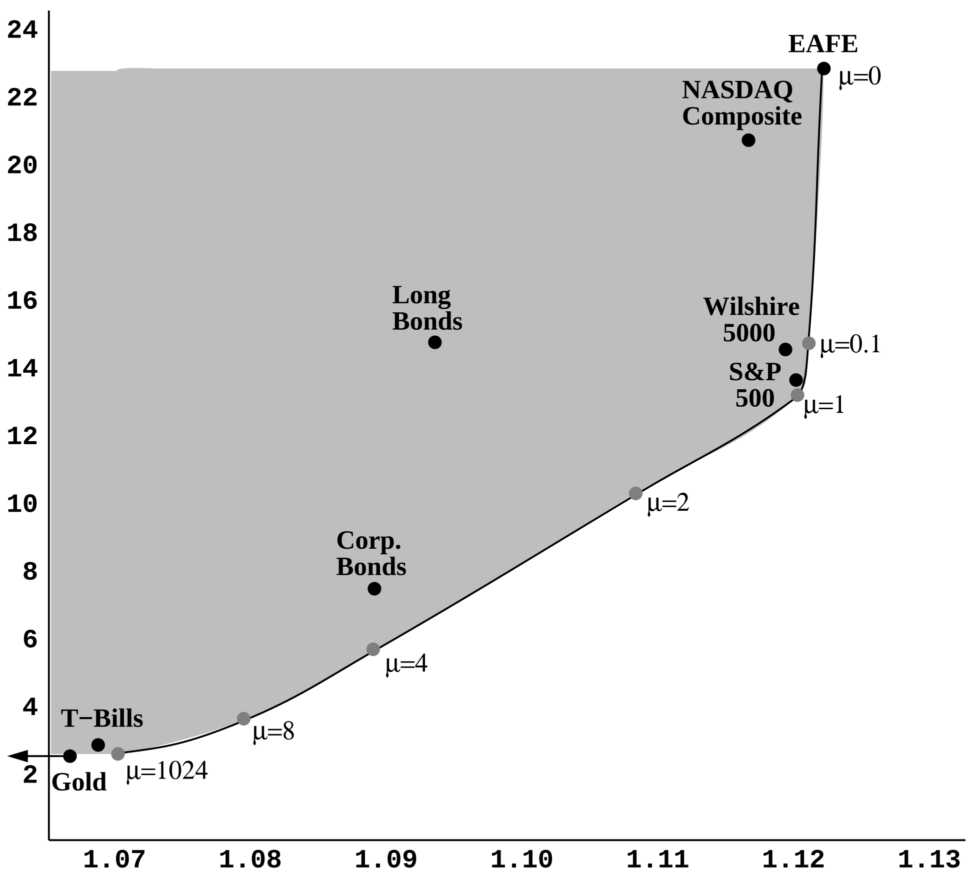
Efficient Frontier

Varying μ produces the so-called efficient frontier.

Portfolios on the efficient frontier are reasonable.

Portfolios not on the efficient frontier can be strictly improved.

μ	Gold	US 3-Month T-Bills	Lehman Bros. Corp. Bonds	NASDAQ Composite	S&P 500	EAFE	Mean	Std. Dev.
0.0						1.000	1.122	0.227
0.1					0.603	0.397	1.121	0.147
1.0					0.876	0.124	1.120	0.133
2.0		0.036	0.322		0.549	0.092	1.108	0.102
4.0		0.487	0.189		0.261	0.062	1.089	0.057
8.0		0.713	0.123		0.117	0.047	1.079	0.037
1024.0	0.008	0.933	0.022	0.016		0.022	1.070	0.028



AMPL Model

```
set A;              # asset categories
set T := {1973..1994};  # years

param lambda default 200;

param R {T,A};

param mean {j in A}
    := ( sum{i in T} R[i,j] )/card(T);

param Rtilde {i in T, j in A}
    := R[i,j] - mean[j];

param Cov {j in A, k in A}
    := sum {i in T} (Rtilde[i,j]*Rtilde[i,k]) / card(T);

var x{A} >=0;

minimize lin_comb:
    lambda *
    sum{i in T} (sum{j in A} Rtilde[i,j]*x[j])^2 / card{T}
    -
    sum{j in A} mean[j]*x[j]
    ;

subject to tot_mass:
    sum{j in A} x[j] = 1;

data;

set A :=
    US_3-MONTH_T-BILLS US_GOVN_LONG_BONDS SP_500 WILSHIRE_5000 NASDAQ_COMPOSITE
    LEHMAN_BROTHERS_CORPORATE_BONDS_INDEX EAFE GOLD;

param R:
US_3-MONTH_T-BILLS US_GOVN_LONG_BONDS SP_500 WILSHIRE_5000 NASDAQ_COMPOSITE
LEHMAN_BROTHERS_CORPORATE_BONDS_INDEX EAFE GOLD :=
1973   1.075   0.942   0.852   0.815   0.698   1.023   0.851   1.677
1974   1.084   1.020   0.735   0.716   0.662   1.002   0.768   1.722
1975   1.061   1.056   1.371   1.385   1.318   1.123   1.354   0.760
    .
    .
    .
1994   1.045   0.889   1.012   0.999   0.968   0.965   1.078   0.990
;

solve;

printf: "-------------------------------------------------------------------\n";
printf: "                                      Asset       Mean    Variance \n";
printf {j in A}: "%45s %10.7f %10.7f \n",
    j, mean[j], sum{i in T} Rtilde[i,j]^2 / card(T);
```

```
printf: "\n";
printf: "Optimal Portfolio:                    Asset      Fraction \n";
printf {j in A: x[j] > 0.001}: "%45s %10.7f \n", j, x[j];

printf: "Mean = %10.7f, Variance = %10.5f \n",
    sum{j in A} mean[j]*x[j],
    sum{i in T} (sum{j in A} Rtilde[i,j]*x[j])ˆ2 / card(T);
```

Interior-Point Methods for Quadratic Programming

Start with an optimization problem—in this case QP:

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

Use slack variables to make all inequality constraints into nonnegativities:

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax - w = b \\ & x, w \geq 0 \end{array}$$

Replace nonnegativity constraints with logarithmic barrier terms in the objective:

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x - \mu \sum_j \log x_j - \mu \sum_i \log w_i \\ \text{subject to} & Ax - w = b \end{array}$$

Introduce Lagrange multipliers to form Lagrangian:

$$c^T x + \frac{1}{2} x^T Q x - \mu \sum_j \log x_j - \mu \sum_i \log w_i - y^T (Ax - w - b)$$

Set derivatives to zero:

$$\begin{aligned} c + Qx - \mu X^{-1} e - A^T y &= 0 \\ -\mu W^{-1} e + y &= 0 \\ -Ax + w + b &= 0 \end{aligned}$$

Introduce dual complementary variables:

$$z = \mu X^{-1} e$$

Rewrite system:

$$\begin{aligned} c + Qx - z - A^T y &= 0 \\ XZe &= \mu e \\ WYe &= \mu e \\ b - Ax + w &= 0 \end{aligned}$$

Introduce step directions: Δx, Δy, Δw, Δz.

Write the above equations for $x + \Delta x$, $y + \Delta y$, $w + \Delta w$, and $z + \Delta z$:

$$\begin{aligned} c + Q(x + \Delta x) - (z + \Delta z) - A^T (y + \Delta y) &= 0 \\ (X + \Delta X)(Z + \Delta Z)e &= \mu e \\ (W + \Delta W)(Y + \Delta Y)e &= \mu e \\ b - A(x + \Delta x) + (w + \Delta w) &= 0 \end{aligned}$$

Rearrange with "delta" variables on left and drop nonlinear terms on left:

$$\begin{aligned} Q\Delta x - \Delta z - A^T \Delta y &= -c - Qx + z + A^T y \\ Z\Delta x + X\Delta z &= \mu e - ZXe \\ W\Delta y + Y\Delta w &= \mu e - WYe \\ -A\Delta x + \Delta w &= -b + Ax - w \end{aligned}$$

This is a **linear** system of $2m + 2n$ equations in $2m + 2n$ unknowns.

Solve'em.

Yadda, yadda, yadda.

Parting Comments

The matrix Q must be positive semidefinite (psd), which means that all of the eigenvalues of Q must be nonnegative.

Such QPs are called convex quadratic programming problems.

Nonconvex QPs are as hard to solve as integer programs. Many local optima, which is best? Hmmm.

The Markowitz model is a convex QP (the covariance matrix is always psd).